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THE QUASITOPOS HULL OF THE CATEGORY OF UNIFORM SPACES

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The quasitopos hull of a concrete category $\mathcal{K}$ is the least cartesian closed topological category with representable strong partial morphisms, containing $\mathcal{K}$ as a dense subcategory. The quasitopos hull of the category of uniform spaces is described: its objects are submetrizable bornological merotopic spaces, i.e., merotopic spaces endowed with a collection of 'bounded' sets related to the merotopy which is, in addition, generated by partial pseudometrics.

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Introduction

We are going to describe the least extension of the category Unif (of uniform spaces) to a cartesian closed topological category with representable strong partial morphisms, i.e., to a concrete quasitopos. The existence of such a least extension, called the quasitopos hull, was established by Wyler [12] who also described the quasitopos hull of the category of topological spaces: it is the category of Choquet (=pseudotopological) spaces. We show that the quasitopos hull of **Unif** is a certain category of bornological merotopic spaces – this is closely related to the cartesian closed topological hull of **Unif** identified as the category of bornological uniform spaces, see [3].

1. Throughout the paper we work with *topological c-categories* in the sense of [5], i.e., with concrete categories $\mathcal{K}$ over Set such that

(i) each structured sink in $\mathcal{K}$ has a final lift,

(ii) for each set X the fibre (i.e., the collection of all $\mathcal{K}$-objects with underlying set X) is small, and

(iii) the fibre of a singleton set is a singleton set.

For terminology see [2] or [5].

In case $\mathcal{K}$ is cartesian closed, condition (iii) is equivalent to each power object L^K having underlying set hom(K, L) (for all $K, L \in \mathcal{K}$), see [4].

2. Let $\mathcal{K}$ be a cartesian closed topological category. Then $\mathcal{K}$ is a quasitopos in the sense of [11] iff strong partial morphisms are representable in $\mathcal{K}$. This is equivalent, as proved in [1], to each object K being an initial subobject of its *one-point extension* K^* defined as follows. Let X be the underlying set of K and let $\infty \notin X$, then K^* is the final lift of the structured sink of all $A \xrightarrow{f} X \cup \{\infty\}$, where $A \in \mathcal{K}$ and f is a map whose restriction defines a $\mathcal{K}$-morphism from the initial subobject of A on the set $f^{-1}(X)$ into K. Assuming K is an initial (=strong) subobject of K^*, then the mono $K \hookrightarrow K^*$ represents strong partial morphisms $A \to K$ i.e., morphisms $f: A_0 \to K$ where $A_0 \hookrightarrow A$ is an initial subobject of A: we extend f to all of A by $f^*(x) = \infty$ whenever $f(x)$ is not defined, then $f^*: A \to K^*$ is the unique morphism such that the following square is a pullback.

$$\begin{tikzcd} A_0 \arrow[r, hook] \arrow[d, "f"'] & A \arrow[d, "f^*"] \\ K \arrow[r, hook] & K^* \end{tikzcd}$$

3. By a *quasitopos hull* of a topological *c*-category $\mathcal{K}$ is meant the least finitely continuous extension (i.e., full and concrete functor) into a topological category which is a quasitopos.

Examples. (i) [12] The quasitopos hull of Top is the category of pseudotopological spaces.

(ii) [1] The quasitopos hull of the category of finite topological spaces (or, of the category of posets) is the category of reflective binary relations.

(iii) [13] The quasitopos hull of the category of supertopological spaces of Doitchinov is the category of superspaces.

Remark. The examples of Wyler are based on the concept of Grothendieck topology. For each Grothendieck topology on $\mathcal{K}$, the sheaves form a quasitopos extension of $\mathcal{K}$. The canonical (=largest) Grothendieck topology then yields the least extension, i.e., the quasitopos hull. Our approach is different, we use the following proposition:

4. Proposition [1]. *Let $\mathcal{L}$ be a concrete quasitopos, and let $\mathcal{K}$ be a full, concrete subcategory of $\mathcal{L}$ which is finally dense (i.e., each $\mathcal{L}$-object is a final lift of a sink with domains in $\mathcal{K}$). Then the quasitopos hull of $\mathcal{K}$ is the full subcategory of $\mathcal{L}$ over all objects which are initial lifts of sources with codomains of the form $(L^K)^*$, where $K, L \in \mathcal{K}$.*

Remark. The CCT (cartesian closed topological) hull has a similar description presented in [7]. Let $\mathscr{L}$ be a CCT category, and let $\mathscr{K}$ be a full, concrete and finally dense subcategory of $\mathscr{L}$. Then the CCT hull of $\mathscr{K}$ is the full subcategory of $\mathscr{L}$ over all objects which are initial lifts of sources with codomains L^K for $K, L \in \mathscr{K}$.

5. Recall that a topological category $\mathscr{K}$ is said to have *quotients stable under pullbacks* if in each pullback opposite a quotient (=final surjective morphism = regular epi) there is a quotient. $\mathscr{K}$ has [finite] *coproducts stable under pullbacks* if for each morphism $f: A \to \coprod_{i\in I} B_i$ [with I finite] we have $A = \coprod_{i\in I} f^{-1}(B_i)$, with coproduct injections obtained by pulling those of $\coprod B_i$ back along f. A topological category is a quasitopos iff it has both quotients and coproducts stable under pullbacks, see [5].

6. In [3] we have introduced the following concrete category $A\mathscr{K}$ for each topological c-category $\mathscr{K}$. Objects are pairs $(X, \mathscr{A})$ where X is an underlying set and $\mathscr{A}$ is a collection of $\mathscr{K}$-objects A with $|A| \subseteq X$ such that

(i) $X = \bigcup_{A\in\mathscr{A}} |A|$, and

(ii) $A \in \mathscr{A}$ implies $B \in \mathscr{A}$ for all $B \sqsubseteq A$, where $B \sqsubseteq A$ means that $|B| \subseteq |A|$ and the inclusion map is a morphism from B to A in $\mathscr{K}$.

Morphisms $f: (X, \mathscr{A}) \to (Y, \mathscr{B})$ of $A\mathscr{K}$ are maps $f: X \to Y$ such that for each $A \in \mathscr{A}$ there is $B \in \mathscr{B}$ such that $f(|A|) \subseteq |B|$ and the domain-range restriction of f is a $\mathscr{K}$-morphism $A \to B$. We proved that if $\mathscr{K}$ has productive quotients, then $A\mathscr{K}$ is cartesian closed. $\mathscr{K}$ is considered as a full subcategory of $A\mathscr{K}$ by identifying each K with $(|K|, \{A \in \mathscr{K} | A \sqsubseteq K\})$.

7. Proposition. *For each topological category $\mathscr{K}$, the category $A\mathscr{K}$ is a quasitopos* iff *$\mathscr{K}$ has quotients stable under pullbacks.*

Proof. (i) Let $\mathscr{K}$ have quotients stable under pullbacks. By forming pullbacks along product projections, we conclude that quotients are finitely productive. Therefore, as proved in [3], $A\mathscr{K}$ is cartesian closed: the power object $(Y, \mathscr{B})^{(X,\mathscr{A})}$ is the object $(\hom((X, \mathscr{A}), (Y, \mathscr{B})), \mathscr{D})$ where $\mathscr{D}$ is the set of all objects D with $|D| \subseteq \hom((X, \mathscr{A}), (Y, \mathscr{B}))$ such that for each $A \in \mathscr{A}$ there is $B \in \mathscr{B}$ for which the evaluation map is a morphism $\mathrm{ev}: D \times A \to B$, $\mathrm{ev}(f, a) = f(a)$.

To prove that $A\mathscr{K}$ has representable strong partial morphisms, for each $(X, \mathscr{A})$ in $A\mathscr{K}$ denote by $\mathscr{A}^{\boldsymbol{\cdot}}$ the collection of all objects $A^{\boldsymbol{\cdot}}$ on subsets of $X \cup \{\infty\}$ for which the initial subobject on $|A^{\boldsymbol{\cdot}}| \cap X$ lies in $\mathscr{A}$. Then $(X, \mathscr{A})$ is clearly the initial subobject of $(X \cup \{\infty\}, \mathscr{A}^{\boldsymbol{\cdot}})$. It is sufficient to show that given an object $(Y, \mathscr{B})$ and a map $f: Y \to X \cup \{\infty\}$, then $f: (Y, \mathscr{B}) \to (X \cup \{\infty\}, \mathscr{A}^{\boldsymbol{\cdot}})$ is a morphism whenever its domain-range restriction is a morphism from $(f^{-1}(X), \mathscr{B}_0)$, the initial subobject of $(Y, \mathscr{B})$, into $(X, \mathscr{A})$. It then follows that $(X \cup \{\infty\}, \mathscr{A}^{\boldsymbol{\cdot}})$ is the one-point extension of $(X, \mathscr{A})$. In fact, for each $B \in \mathscr{B}$ we denote by $A^{\boldsymbol{\cdot}}$ the final object of the domain-range restriction $f_0: |B| \to f(|B|)$ of f, and it is sufficient to show that $A^{\boldsymbol{\cdot}} \in \mathscr{A}^{\boldsymbol{\cdot}}$. This follows from the fact that, since $f_0: B_0 \to A$ is a quotient, the pullback along the inclusion of the subobject A of $A^{\bullet}$ on the set $|A^{\bullet}| \cap X$ yields a quotient:

$$\begin{tikzcd} B_0 \arrow[r, hook] \arrow[d, "f_0"'] & B \arrow[d, "f_1"] \\ A \arrow[r, hook] & A^{\bullet} \end{tikzcd}$$

We have $B_0 \in \mathscr{B}_0$, and since f_1 is also a domain-range restriction of f, we conclude that $A \in \mathscr{A}$, and hence $A^{\bullet} \in \mathscr{A}^{\bullet}$.

(ii) Let $A\mathscr{K}$ be a quasitopos. Then $A\mathscr{K}$ has quotients stable under pullbacks, and hence, it is sufficient to verify that $\mathscr{K}$ is closed under pullbacks and quotients in $A\mathscr{K}$. The first follows from the fact that $\mathscr{K}$ is finally dense in $A\mathscr{K}$ (each $(X, \mathscr{A})$ is the final lift of all $A \in \mathscr{A}$), and hence, closed under initial lifts. The latter is clear: if $e: K \to L$ is a quotient in $\mathscr{K}$, then for each morphism $e: K \to (Y, \mathscr{B})$ in $A\mathscr{K}$ we have $L \in \mathscr{B}$, and hence $\{B \mid B \sqsubseteq L\} \subseteq \mathscr{B}$. $\square$

8. Remark. Let $\mathscr{K}$ be a topological category which has both quotients and finite coproducts stable under pullbacks. Then the full subcategory $A^*\mathscr{K}$ of $A\mathscr{K}$ formed by all $(X, \mathscr{A})$ with $\mathscr{A}$ directed (i.e., for $A_1, A_2 \in \mathscr{A}$ there is always $A \in \mathscr{A}$ with $A_1 \sqsubseteq A$ and $A_2 \sqsubseteq A$) is a quasitopos.

In fact, in the preceding proof, if $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are both directed, then so is $(\hom((X, \mathscr{A}), (Y, \mathscr{B})), \mathscr{D})$: given $D_1, D_2 \in \mathscr{D}$, the natural quotient D of $D_1 + D_2$ whose underlying set is a subset of $(\hom(X, \mathscr{A}), (Y, \mathscr{B}))$ is an element of $\mathscr{D}$ because for each $A \in \mathscr{A}$, we have $D \times A$ as a quotient of $(D_1 + D_2) \times A = D_1 \times A + D_2 \times A$. Furthermore, for each directed $(X, \mathscr{A})$, $(X \cup \{\infty\}, \mathscr{A}^{\bullet})$ is also directed: given $D_1, D_2 \in \mathscr{A}^{\bullet}$ consider again the natural quotient D of $D_1 + D_2$ with the underlying set in $X \cup \{\infty\}$, then $D \in \mathscr{A}^{\bullet}$.

Example. The category **Merot** of merotopic spaces has both quotients and finite coproducts stable under pullbacks, and hence, A^***Merot** is a quasitopos.

Merotopic spaces were introduced by Katětov [10] via micromeric collections, but the category **Merot** can be identified with that of semiuniform spaces of Isbell [9], i.e., of pairs (X, α) where α is a filter of covers of X; see also [6].

9. Remark. In [3], the cartesian closed topological hull of **Unif** was described as the following category **BUnif** of *bornological uniform spaces.* Objects are triples $(X, \beta, \mathscr{B})$ where β is a uniformity on the set X and $\mathscr{B}$ is a bornology (i.e., a collection of 'bounded' subsets containing all singleton sets and closed under subsets and finite unions) satisfying the following conditions:

(a) β is $\mathscr{B}$-final, i.e., the space (X, β) is the final lift of its bounded subspaces,

(b) $\mathscr{B}$ is β-closed, i.e., a set $M \subseteq X$ is bounded whenever for each cover $\mathscr{U} \in \beta$ there is a bounded set whose $\mathscr{U}$-star contains M.

Morphisms are bounded, uniformly continuous maps. Each uniform space (X, β) is identified with the object $(X, \beta, \exp X)$, and thus, **Unif** is considered as a full subcategory of **BUnif**.

By substituting uniformity with merotopy in the above definition, we obtain the following concept:

10. Definition. A *bornological merotopic space* is a triple $(X, \beta, \mathcal{B})$ where β is a merotopy and $\mathcal{B}$ is a bornology satisfying (a) and (b) above.

Remark. Each bornological merotopic space (X, β) can be identified with the object $(X, \mathcal{A})$ of A^***Merot** where $A \in \mathcal{A}$ iff A is finer than some bounded subspace of (X, β). In this sense, we regard the category **BMerot** of bornological merotopic spaces as a full subcategory of A^***Merot.**

11. Definition. A bornological merotopic space $(X, \beta, \mathcal{B})$ is said to be *submetrizable* if there is a collection r_i of pseudometrics defined on sets $X_i \subseteq X (i \in I)$ with the following properties:

(i) β is the coarsest $\mathcal{B}$-final merotopy with each r_i uniformly continuous on the subspace $X_i (i \in I)$ and

(ii) $\mathcal{B}$ contains each set $M \subseteq X$ such that for each $i \in I$ and $\varepsilon > 0$ there is $N \in \mathcal{B}$ such that $r_i(x, N \cap X_i) < \varepsilon$ for every $x \in M \cap X_i$.

Remarks. (i) Condition (a) means that β is the $\mathcal{B}$-final modification of the merotopy with the subbase formed by covers $\mathcal{U}(r_i, \varepsilon)$ for $i \in I$ and $\varepsilon > 0$ where $\mathcal{U}(r_i, \varepsilon)$ consists of sets $\{y \in X | \text{if } y \in X_i \text{ then } r_i(x, y) < \varepsilon\}$ for $x \in X_i$.

(ii) Each bornological uniform space is submetrizable - consider the collection of all uniformly continuous pseudometrics defined on all of the space.

We denote by **SBMerot** the category of submetrizable bornological merotopic spaces. We then have full concrete subcategories

$$\mathbf{Unif} \subseteq \mathbf{BUnif} \subseteq \mathbf{SBMerot} \subseteq \mathbf{BMerot} \subseteq A^*\mathbf{Merot}.$$

12. Theorem. *The category* **SBMerot** *of submetrizable bornological merotopic spaces is the quasitopos hull of* **Unif.**

Proof. We apply Proposition 4 on the quasitopos A^***Merot**, and we shall prove that

(a) **Unif** is finally dense in A^***Merot**, and

(b) the initial hull of the objects $(P^Q)^*$ for uniform spaces P, Q in A^***Merot** is the category **SBMerot**.

For (a), observe that $\mathcal{K}$ is always dense in $A\mathcal{K}$, and hence, **Merot** is finally dense in A^***Merot**. Thus, it is sufficient to prove that **Unif** is finally dense in **Merot**.

We shall verify that each merotopic space (X, α) is a quotient of a uniform space. Put $\alpha = \{\mathcal{U}_i | i \in I\}$, and consider each cover $\mathcal{U}_i$ as a discrete uniform space (the points of which are members of $\mathcal{U}_i$). Let $\tilde{A}$ denote the indiscrete uniform space on the set X, and let B be the subspace of $A\times\prod_{i\in I}\mathcal{U}_i$ on the set of all points $(x,(U_i)_{i\in I})$ with $x\in\bigcap_{i\in I}U_i$ and $U_i\in\mathcal{U}_i$ for each $i\in I$. Each cover $\mathcal{U}_j\in\alpha$ defines the following uniform cover $\mathcal{U}_j^*=\{U^*|U\in\mathcal{U}_j\}$ of B:

$$U^*=B\cap\left(U\times\prod_{i\in I}\mathcal{U}_i'\right),\qquad \mathcal{U}_i'=\mathcal{U}_i\quad\text{if } i\neq j \text{ and } \mathcal{U}_j'=\{U\}.$$

Moreover, the covers

$$\mathcal{U}_{j_1}^*\wedge\mathcal{U}_{j_2}^*\wedge\cdots\wedge\mathcal{U}_{j_n}^*,\quad j_1,j_2,\ldots,j_n\in J,$$

form a base of the uniformity of B. It follows that the first projection $(x,(U_i)_{i\in I})\to x$ defines a quotient map $e:B\to A$. In fact,

(i) e is continuous because for each $\mathcal{U}_j\in\alpha$ the cover $\mathcal{U}_j^*$ refines $e^{-1}[\mathcal{U}_j]$;

(ii) e is final because for each $\mathcal{U}_j^*$ we have $e[\mathcal{U}_j^*]=\mathcal{U}_j$, and hence, for each uniform cover $\mathcal{V}$ of B we have $e[\mathcal{V}]\in\alpha$.

Before proving (b), let us remark that (a) guarantees that each full concrete subcategory $\mathcal{L}$ of A^***Merot** containing **Unif** is closed in A^***Merot** under formation of initial lifts and power objects. The statement (b) will be proved by verifying the following:

(b1) Each submetrizable bornological merotopic space is an initial lift of a source with codomains P^Q for uniform spaces P, Q;

and the category **BSMerot** is closed in A^***Merot** under

(b2) initial lifts,

(b3) the passage $P\to P^\#$, and

(b4) the formation of hom-objects of uniform spaces.

First, in order to prove (b1), it is sufficient to find, for each submetrizable space R, an initial source $f_j:R\to R_j^\#$ $(j\in J)$ with each R_j in **BUnif**. In fact, since **BUnif** is the CCT hull of **Unif**, for each $j\in J$ we then have an initial source $g_{ji}:R_j\to P_{ji}^{Q_{ji}}(i\in I_j)$ with P_{ji} and Q_{ji} uniform spaces. It is easy to see that the source $g_{ji}^\#:R_j^\#\to(P_{ji}^{Q_{ji}})^\#$ is also initial, and hence, we get the required initial source by composition: $g_{ji}f_j:R\to(P_{ji}^{Q_{ji}})^\#$. Thus, let $R=(X,\beta,\mathcal{B})$ be a submetrizable space with respect to a collection r_i of pseudometrics defined on $X_i\subseteq X(i\in I)$. For each $i\in I$ denote by $\mathcal{B}_i$ the bornology on the set X_i which consists of all $M\subseteq X_i$ satisfying the following condition:

(c_i) for each $\varepsilon>0$ there exists $N\in\mathcal{B}$, $N\subseteq X_i$ with $r_i(x,N)<\varepsilon$ for every $x\in M$.

Let β_i be the uniformity on X_i obtained as the $\mathcal{B}_i$-final modification of the uniformity induced by r_i. Then $(X_i,\beta_i,\mathcal{B}_i)$ is a bornological uniform space. We shall prove that the source

$$f_i:\ (X,\beta,\mathcal{B})\to(X_i,\beta_i,\mathcal{B}_i)^\#,\quad i\in I,$$

with

$$f_i(x)=\begin{cases}x & \text{for } x\in X_i,\\ \infty & \text{for } x\in X-X_i\end{cases}$$

is initial. In fact, each f_i is a morphism because its domain-restriction to the subobject

$f_i^{-1}(X_i)=X_i$ of $(X, \beta, \mathscr{B})$ is an (identity carried) morphism into $(X_i, \beta_i, \mathscr{B}_i)$. Moreover, β is the coarsest $\mathscr{B}$-final merotopy for which those restrictions are uniformly continuous for all $i \in I$, i.e., the coarsest among $\mathscr{B}$-final merotopies γ such that for each i the subspace $(X_i, \gamma/X_i)$ is finer than (X_i, β_i). (In fact, each such γ has the property that given $M \in \mathscr{B}$, $M \subseteq X_i$, then r_i is uniformly continuous on $(M, \gamma/M)$ - because $M \in \mathscr{B}_i$, and β_i/M is the uniformity induced by r_i - consequently, γ/M is finer than β/M for each $M \in \mathscr{B}$, and hence, γ is finer than β.) Moreover, $\mathscr{B}$ is the largest β-closed bornology for which the above restrictions preserve bounded sets since $\mathscr{B}$ contains each set satisfying (c_i) above for all i. Thus, the above source is initial by the initial completeness of **BMerot**.

To prove (b2), consider an initial source

$$f_j\colon\ P \to (Y_j, \beta_j, \mathscr{B}_j) \in \textbf{SBMerot}\ (j \in J)$$

in A^***Merot**. Then P is clearly the bornological merotopic space $(X, \beta, \mathscr{B})$ with

$$\mathscr{B} = \{M \mid M \subseteq X, f_j(M) \in \mathscr{B}_j \text{ for each } j \in J\}$$

and β is the $\mathscr{B}$-final modification of the initial merotopy (having subbase $f_j^{-1}[\mathscr{U}]$ for $j \in J$, $\mathscr{U} \in \beta_j$). Moreover, for each j there is a collection r_{ji} of pseudometrics defined on sets $Y_{ji} \subseteq Y_j (i \in I_j)$ such that β_j is the $\mathscr{B}_j$-final modification of the coarsest merotopy β_j' making r_{ji} uniformly continuous (i.e., having subbase formed by all $\mathscr{U}(r_{ji}, \varepsilon)$ for $i \in I_j$ and $\varepsilon > 0$).

Define pseudometrics $\bar{r}_{ji}$ on $\bar{X}_{ji} = f^{-1}(Y_{ji})$ by $\bar{r}_{ji}(a, b) = r_{ji}(f_j(a), f_j(b))$, and let β' be the merotopy the subbase of which is formed by all the covers $\mathscr{U}(\bar{r}_{ji}, \varepsilon)$ for all j, i and $\varepsilon > 0$. Since clearly $\mathscr{U}(\bar{r}_{ji}, \varepsilon) \subset f_j^{-1}(\mathscr{U}(r_{ji}, \varepsilon))$ and $f_j^{-1}(\mathscr{U}(r_{ji}, \varepsilon))$ is finer than $\mathscr{U}(\bar{r}_{ji}, 2\varepsilon)$, we conclude that β is the $\mathscr{B}$-final modification of β'. This proves (i) of Definition 11. To prove (ii), let M have the property of (ii), then for each j, i, $\varepsilon > 0$ we have $N \in \mathscr{B}$ with $\bar{r}_{ji}(x, N \cap \bar{X}_{ji}) < \varepsilon$ for all $x \in M \cap \bar{X}_{ji}$. This implies that $r_{ji}(y, f_j(N) \cap Y_{ji}) < \varepsilon$ for all $y \in f_j(M) \cap Y_{ji}$ and since $f_j(N) \in \mathscr{B}_j$, we conclude that $f_j(M) \in \mathscr{B}_j$ for each j. Thus, $M \in \mathscr{B}$ and (ii) holds.

The proof of (b3) is elementary: for each $(Y, \beta, \mathscr{B})$ in **SBMerot** we have $(Y, \beta, \mathscr{B})^{\#} = (Y \cup \{\infty\}, \beta^{\bullet}, \mathscr{B}^{\bullet})$ where $\beta^{\bullet}$ consists of all covers $\mathscr{U}^{\bullet} = \{U \cup \{\infty\} \mid U \in \mathscr{U}\}$ for $\mathscr{U} \in \beta$, and $\mathscr{B}^{\bullet}$ consists of all $M \subseteq Y \cup \{\infty\}$ with $M \cap Y \in \mathscr{B}$. The same pseudometrics used in Definition 11 for $(Y, \beta, \mathscr{B})$ demonstrate that $(Y, \beta, \mathscr{B})^{\#}$ is a submetrizable bornological merotopic space.

Finally, (b4) follows from the fact that **BUnif**, which is the CCT hull of **Unif**, is contained in **SBMerot**. □

Open problem. Characterize the quasitopos hull of **Merot**.

Remark. This hull can be found as a full subcategory of A^***Merot**, but the question is what subcategory.

In the course of the International Conference on Categorical Topology (L'Aquila, Italy, 1986) we have announced a characterization of the quasitopos hull of Merot. But this was unfortunately wrong.

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